An Analysis of the Quantum Penny Flip Game using Geometric Algebra

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We analyze the quantum penny flip game using geometric algebra and so determine all possible unitary transformations which enable the player Q to implement a winning strategy. Geometric algebra provides a clear visual picture of the quantum game and its strategies, as well as providing a simple and direct derivation of the winning transformation, which we demonstrate can be parametrized by two angles θ, ϕ. For comparison we derive the same general winning strategy by conventional means using density matrices.

KEYWORDS: quantum game, penny flip, geometric algebra, rotor, density matrix

1. Introduction

In 1999 Meyer[1] introduced the quantum version of the penny flip game, a seminal paper for quantum game theory.[2–17] In the classical form of this game a coin is placed heads up inside a box so that the state of the coin is hidden from the players. The first player Q then either flips the coin or leaves it unchanged, following which the second player P also either flips the coin or not, and finally Q flips the coin or not, after which the coin is inspected. If the coin is heads up Q wins, otherwise P wins. Classically each player has an equal chance of winning and the optimal strategy, in order to prevent each player predicting the other's behaviour, is to randomly flip the coin or not, corresponding to a mixed strategy Nash equilibrium.[18]

In the quantum version of the game P is restricted to classic strategies whereas Q adopts quantum strategies and so is able to apply unitary transformations to the possible states of the coin, which behaves like a spin half particle with a general state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ and $|1\rangle$ are orthonormal states representing heads and tails respectively, and α, β are complex numbers. Meyer identifies a winning strategy for Q as the application of the Hadamard transform, in which case the operation by P has no effect:

$$|0\rangle \xrightarrow{Q} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{P} \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \xrightarrow{Q} |0\rangle. \tag{1}$$

The final Hadamard operation by Q returns the state to the starting position, thus Q wins every game using a quantum strategy. In simple terms we can view the Hadamard operation

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as placing the coin "on its edge", which is why the flip operation of the following player has no effect.

Our aim in this paper is firstly to find the most general unitary transformations which lead to a winning strategy for Q, and secondly to demonstrate that geometric algebra provides a convenient formalism with which to find the general solution, which is parametrized by angles θ, ϕ. Our motivation in using geometric algebra is ultimately to investigate quantum mechanical correlations in strategic interactions between two or more players of quantum games, and more generally to exploit the analytical tools of game theory to better understand quantum correlations. We demonstrate in Section 4, however, that for the quantum penny flip game conventional methods of analysis using density matrices[19] are also effective in analyzing this game, and run parallel to the geometric algebra approach, but we believe that for n-player games the concise formalism of geometric algebra is advantageous.

Geometric algebra[20–23] is a unified mathematical formalism which simplifies the treatment of points, lines, planes in quantum mechanical spin half systems.[24] In general, given a linear vector space V with elements $u, v, \dots$ we may form[25] the tensor product $U \otimes V$ of vector spaces U, V containing elements (bivectors) $u \otimes v$. The vector space may be extended to a vector space $\Lambda(V)$ of elements consisting of multivectors which can be multiplied by means of the exterior (wedge) product $u \wedge v$. The noncommutative geometric product uv of two vectors u, v is defined by $uv = u \cdot v + u \wedge v$, which is the sum of the scalar inner product and the bivector wedge product, and may be extended to the geometric product of any two multivectors.

Properties of the Pauli algebra have previously been developed[24] in the context of geometric algebra. Denote by $\{\sigma_i\}$ an orthonormal basis in $\mathbb{R}^3$, then $\sigma_i \cdot \sigma_j = \delta_{ij}$. We also have $\sigma_i \wedge \sigma_i = 0$ for each $i = 1, 2, 3$ and so in terms of the geometric product we have $\sigma_i^2 = \sigma_i \sigma_i = 1$, and $\sigma_i \sigma_j = \sigma_i \wedge \sigma_j = -\sigma_j \sigma_i$ for each $i \neq j$. Hence the basis vectors anticommute with respect to the geometric product. Denote by ι the trivector

$$\iota = \sigma_1 \sigma_2 \sigma_3, \tag{2}$$

where the associative geometric product $\sigma_1 \sigma_2 \sigma_3$ of a bivector $\sigma_1 \wedge \sigma_2$ and an orthogonal vector σ_3 is defined by

$$\sigma_1 \sigma_2 \sigma_3 = (\sigma_1 \wedge \sigma_2)\sigma_3 = \sigma_1 \wedge \sigma_2 \wedge \sigma_3.$$

We have $\sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_3 \sigma_3 = \iota \sigma_3$ and so $\sigma_i \sigma_j = \iota \sigma_k$ for cyclic i, j, k. We also find by using anticommutativity, associativity, and $\sigma_i^2 = 1$ that $\iota^2 = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 = -1$ and, furthermore, that ι commutes with each vector σ_i. We may summarize the algebra of the basis vectors $\{\sigma_i\}$ by the relations

$$\sigma_i \sigma_j = \delta_{ij} + \iota \varepsilon_{ijk} \sigma_k, \tag{3}$$
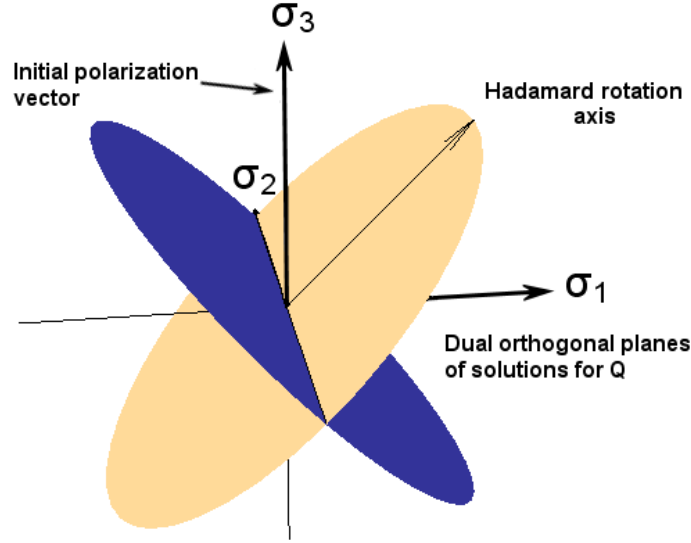
which is isomorphic to the algebra of the Pauli matrices.

Fig. 1.   The two intersecting planes contain the unit vector u which defines the axis of rotation about which Q rotates σ_3 to $\pm\sigma_1$, through an angle θ (color online).

We also require the following well known result in geometric algebra. For any unit vector u we can rotate a vector v by an angle θ in the plane perpendicular to u by applying a rotor R defined by

$$R = e^{\iota\theta u/2} = \cos\frac{\theta}{2} + \iota u \sin\frac{\theta}{2}, \tag{4}$$

which acts according to $v \xrightarrow{R} v' = RvR^\dagger$. R is unitary in the sense that $RR^\dagger = R^\dagger R = 1$, where the $\dagger$ operation acts by inverting the order of terms and flipping the sign of ι, and corresponds to the Hermitean conjugate when acting on the Pauli matrices. The even subalgebra of the geometric algebra of multivectors consists of grade zero and grade two multivectors which correspond to a spinor with four real components. In summary, the spinor algebra of the Pauli matrices and the unitary matrices which rotate a polarization axis as displayed on the Bloch sphere may be analyzed by means of geometric algebra in three dimensions in which vectors are operated on by a rotor.[24]

2.   The Quantum Penny Flip Game using Geometric Algebra

The state of the quantum coin for heads up is $|0\rangle$ which is depicted on the Bloch sphere by the polarization vector pointing up on the σ_3 axis, corresponding to the initial vector $\psi_0 = \sigma_3$ as shown in Figure 1. Following operations performed by Q, P, Q in turn, in which Q always wins, the final wavefunction ψ_3 also corresponds to the unit vector σ_3.

Suppose Q first applies a general unitary transformation, represented by a rotor (4), namely $U_1 = e^{\iota\theta u/2}$ to obtain the state $\psi_1 = U_1\psi_0 U_1^\dagger$. P now applies the optimal classical

strategy of applying a coin flip operation F with probability p and no flip operation N with probability $1 - p$, to obtain the mixed state

$$\psi_2 = pF\psi_1 F^\dagger + (1-p)N\psi_1 N^\dagger = pFU_1\psi_0 U_1^\dagger F^\dagger + (1-p)NU_1\psi_0 U_1^\dagger N^\dagger.$$

The coin flip F is equivalent to the action on the spinor $(|0\rangle, |1\rangle)$ of the Pauli matrix σ_1 which is isomorphic to σ_1 in geometric algebra, so we have simply $F = \sigma_1$ and also $N = 1$. Q now applies a final unitary transformation U_3 which is independent of p to obtain

$$\psi_3 = U_3\psi_2 U_3^\dagger = pU_3\sigma_1 U_1\sigma_3 U_1^\dagger \sigma_1 U_3^\dagger + (1-p)U_3 U_1\sigma_3 U_1^\dagger U_3^\dagger. \tag{5}$$

Since we assume that Q always wins, i.e. $\psi_3 = \sigma_3$ for any p, the terms in this expression must equal $p\sigma_3$ and $(1-p)\sigma_3$ respectively. For the second term this requires $U_3 U_1\sigma_3 U_1^\dagger U_3^\dagger = \sigma_3$ and so $U_3 U_1$ must commute with σ_3. Hence $U_3 U_1 = e^{\iota\phi\sigma_3/2}$ for some angle ϕ, i.e. $U_1 = U_3^\dagger e^{\iota\phi\sigma_3/2}$. On substituting into (5) we find

$$\psi_3 = pU_3\sigma_1 U_3^\dagger \sigma_3 U_3\sigma_1 U_3^\dagger + (1-p)\sigma_3, \tag{6}$$

which has no explicit dependence on the angle ϕ which therefore remains arbitrary. Evidently it is not necessary that U_3 be inverse to the initial rotation U_1, i.e. the final rotation need not be about the same axis as the initial rotation. In order for the first term in (6) to equal $p\sigma_3$ we require $U_3\sigma_1 U_3^\dagger \sigma_3 U_3\sigma_1 U_3^\dagger = \sigma_3$, that is,

$$U_3\sigma_1 U_3^\dagger \sigma_3 = \sigma_3 U_3\sigma_1 U_3^\dagger, \tag{7}$$

and so $U_3\sigma_1 U_3^\dagger$ commutes with σ_3. This implies that $U_3\sigma_1 U_3^\dagger$ is a multiple of σ_3, since the rotated vector $U_3\sigma_1 U_3^\dagger$ is a linear combination of the basis elements, i.e. $U_3\sigma_1 U_3^\dagger = c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3$ for some scalars c_i, and the Pauli algebra (3) then implies that (7) is satisfied only if $c_1 = c_2 = 0$. Since $U_3\sigma_1 U_3^\dagger$ is a unit vector we also have $c_3 = \pm 1$.

The final state ψ_3 is therefore equal to σ_3, namely heads up independent of p, provided $U_3 = e^{\iota\phi\sigma_3/2}U_1^\dagger$ and $U_1\sigma_3 U_1^\dagger = \pm\sigma_1$. Hence Q's strategy is clear: by rotating the starting vector σ_3 to $\pm\sigma_1$, P's coin flip operation has no effect because $F\sigma_1 F^\dagger = \sigma_1\sigma_1\sigma_1 = \sigma_1$, and so Q simply then applies $U_3 = e^{\iota\phi\sigma_3/2}U_1^\dagger$ to turn the coin back to heads where it started.

3. Solution for Q's Winning Strategy

By substituting for the general rotor $U_1 = R$ as given in (4), and by writing the unit vector as $u = a\sigma_1 + b\sigma_2 + c\sigma_3$ where the scalars a, b, c satisfy $a^2 + b^2 + c^2 = 1$, we find that we require $R^\dagger\sigma_1 = c_3\sigma_3 R^\dagger$, specifically

$$\left[\cos\frac{\theta}{2} - \iota\sin\frac{\theta}{2}(a\sigma_1 + b\sigma_2 + c\sigma_3)\right]\sigma_1 = c_3\sigma_3\left[\cos\frac{\theta}{2} - \iota\sin\frac{\theta}{2}(a\sigma_1 + b\sigma_2 + c\sigma_3)\right], \tag{8}$$

where $c_3^2 = 1$. This equation is satisfied if and only if $a\sin\frac{\theta}{2} = c_3 c\sin\frac{\theta}{2}$ and $b\sin\frac{\theta}{2} = c_3\cos\frac{\theta}{2}$, which implies $\sin\frac{\theta}{2} \neq 0$. Hence $a = c_3 c$ and $b = c_3\cot\frac{\theta}{2}$. Since u is a unit vector we have $2a^2 + \cot^2\frac{\theta}{2} = 1$ which implies $|\cot\frac{\theta}{2}| \leqslant 1$, and hence θ can take any value such that $\frac{\pi}{2} \leqslant$

$|\theta| \leqslant \frac{3\pi}{2}$. Then

$$a = \pm\sqrt{\frac{1}{2} - \frac{1}{2}\cot^2\frac{\theta}{2}}\,, \tag{9}$$

together with $c = c_3 a$ and $b = c_3 \cot\frac{\theta}{2}$ where $c_3 = \pm 1$. Thus we have the general expression

$$U_1 = e^{\iota\theta(a\sigma_1 + c_3\cot\frac{\theta}{2}\sigma_2 + c_3 a\sigma_3)/2}, \tag{10}$$

with which Q rotates σ_3 to $\pm\sigma_1$ about the axis defined by u through the angle θ, for any θ in the specified range. The unit vector $u = a\sigma_1 + b\sigma_2 + c\sigma_3$ lies in one of the two intersecting planes defined by $|a| = |c|$, as shown in Figure 1. Denote the angle between u and σ_3 by ψ then $\cos\psi = \frac{1}{2}(\sigma_3 u + u\sigma_3) = c = \pm a$ which implies $-1/\sqrt{2} \leqslant \cos\psi \leqslant 1/\sqrt{2}$, showing that u is tilted with respect to σ_3 at an angle ψ in the range $\pi/4 \leqslant |\psi| \leqslant 3\pi/4$.

The choice of sign for a in (9) can in effect be altered by replacing $\theta \to -\theta$ in (10), and the sign $c_3 = \pm 1$ can be reversed by replacing $\sigma_2 \to -\sigma_2, \sigma_3 \to -\sigma_3$, which leaves the Pauli algebra (3) invariant, and may be implemented by rotating the system about the σ_1 axis through π by means of the rotor $S = e^{\iota\pi\sigma_1/2} = \iota\sigma_1$. The final move by Q is the rotation $U_3 = e^{\iota\phi\sigma_3/2}U_1^\dagger$ which depends on two parameters θ, ϕ, where $e^{\iota\phi\sigma_3/2}$ performs a rotation about the σ_3 axis leaving σ_3 unchanged.

We recover Meyer's solution by choosing $\theta = \pi, \phi = 0$ together with appropriate signs, to obtain $U_1 = e^{\iota\frac{\pi}{2}(\sigma_1 + \sigma_3)/\sqrt{2}} = U_3^\dagger$, which performs a rotation of $\theta = \pi$ about the line defined by the vector $(\sigma_1 + \sigma_3)/\sqrt{2}$, and so reproduces the Hadamard transform which rotates the polarization vector onto the σ_1 axis as shown in Figure 1.

4. Analysis using Density Matrices

The analysis of the quantum penny flip game using geometric algebra can be reproduced by means of density matrices and unitary transformations.[19] We may write any 2×2 unitary matrix in the form $U = e^{iA}$ where the Hermitean matrix A can be expanded in terms of the Pauli matrices σ_i and the identity matrix I_2 according to $A = \alpha(a\sigma_1 + b\sigma_2 + c\sigma_3) + \beta I_2$ where the scalars a, b, c are normalized such that $a^2 + b^2 + c^2 = 1$, and where α, β are fixed angles. If we define $\theta = 2\alpha$ and also the 2×2 matrix $\widehat{u} = a\sigma_1 + b\sigma_2 + c\sigma_3$ (which satisfies $\widehat{u}^2 = I_2$), then

$$U = e^{i\widehat{u}\theta/2}e^{i\beta} = \left(I_2\cos\frac{\theta}{2} + i\widehat{u}\sin\frac{\theta}{2}\right)e^{i\beta}, \tag{11}$$

which compares with the expression (4) for the rotor R. We emphasize, however, that in (4) the element ι is a tri-vector and u denotes a unit vector which is a linear combination of basis vectors σ_i.

If we denote the starting state by $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, then the first move by Q is to apply a general unitary transformation U_1 on the starting density matrix $\rho_0 = |0\rangle\langle 0|$, which therefore evolves to $\rho_1 = U_1\rho_0 U_1^\dagger$. P now applies the optimal classical strategy of applying a coin flip operation

$F = \sigma_1$ with probability p and the no flip operation $N = I_2$ with probability $1 - p$ producing

$$\rho_2 = pF\rho_1 F^\dagger + (1-p)N\rho_1 N^\dagger = p\sigma_1 U_1 \rho_0 U_1^\dagger \sigma_1 + (1-p)U_1 \rho_0 U_1^\dagger.$$

Q applies a final unitary transformation U_3 which is independent of p to obtain

$$\rho_3 = U_3 \rho_2 U_3^\dagger = pU_3\sigma_1 U_1 \rho_0 U_1^\dagger \sigma_1 U_3^\dagger + (1-p)U_3 U_1 \rho_0 U_1^\dagger U_3^\dagger, \tag{12}$$

which is a matrix equation which can be compared with the geometric algebraic expression (5), in which the unit vector σ_3 replaces the initial density matrix ρ_0 and U_3 implements a quaternion rotation of σ_3. Since we assume that Q always wins, i.e. that $\rho_3 = |0\rangle\langle 0|$ for any p, the terms in the expression (12) must equal $p|0\rangle\langle 0|$ and $(1-p)|0\rangle\langle 0|$ respectively, which for the second term requires $U_3 U_1 |0\rangle\langle 0| U_1^\dagger U_3^\dagger = |0\rangle\langle 0|$, where $U = U_3 U_1$ is a unitary matrix.

The general solution of the matrix equation $U|0\rangle\langle 0|U^\dagger = |0\rangle\langle 0|$ for any 2×2 unitary matrix U, where $|0\rangle\langle 0| = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}(\sigma_3 + I_2)$, may be found by parametrizing U as above or, more directly, observing that this equation is equivalent to $[U, \sigma_3] = 0$. The solution is $U = e^{i\beta}e^{i\sigma_3\phi/2}$ for angles β, ϕ.

Hence we have $U_3 U_1 = e^{i\beta}e^{i\sigma_3\phi/2}$ and on substituting $U_1|0\rangle\langle 0|U_1^\dagger = U_3^\dagger|0\rangle\langle 0|U_3$ into Eq. (12) we find

$$\rho_3 = pU_3\sigma_1 U_3^\dagger|0\rangle\langle 0|U_3\sigma_1 U_3^\dagger + (1-p)|0\rangle\langle 0|, \tag{13}$$

which has no explicit dependence on the angle ϕ which therefore remains arbitrary. In order that the first term in Eq. (13) equal $p|0\rangle\langle 0|$ we require

$$U_3\sigma_1 U_3^\dagger|0\rangle\langle 0|U_3\sigma_1 U_3^\dagger = |0\rangle\langle 0|, \tag{14}$$

where $U = U_3\sigma_1 U_3^\dagger$ is unitary. As discussed above, this matrix equation is equivalent to $[U, \sigma_3] = 0$ which implies that U is a linear combination of I_2 and σ_3. We also have $U^2 = I_2$ which implies, since $U \neq \pm I_2$, that $U = \pm\sigma_3 = U_3\sigma_1 U_3^\dagger$.

Thus the final state is heads up independent of p, provided $U_3 = e^{i\beta}e^{i\sigma_3\phi/2}U_1^\dagger$ and $U_1\sigma_3 U_1^\dagger = \pm\sigma_1$. The phase angle β can be set to zero without loss of generality. By substituting for the general unitary transformation $U_1 = U$ as given by Eq. (11) we require $U^\dagger\sigma_1 = \pm\sigma_3 U^\dagger$, specifically:

$$\left[I_2\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(a\sigma_1 + b\sigma_2 + c\sigma_3)\right]\sigma_1 = \pm\sigma_3\left[I_2\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(a\sigma_1 + b\sigma_2 + c\sigma_3)\right], \tag{15}$$

which compares with the isomorphic Eq. (8) derived using geometric algebra, and which therefore has the solution Eq. (10) in which $\sigma_1, \sigma_2, \sigma_3$ now refer to Pauli matrices, instead of unit vectors.

Evidently this derivation of the general solution closely parallels that using geometric algebra, which uses quaternion rotations of vectors in real 3-space, with the formalism defined in terms of unit vectors $\sigma_1, \sigma_2, \sigma_3$, whereas the density matrix formalism uses Dirac's bra-ket notation, density matrices and complex matrices for $SU(2)$ rotations. Geometric algebra has

the advantage of avoiding global phase factors $\mathrm{e}^{i\beta}$ and also permits a geometric picture as shown in Figure 1, which is hidden in the density matrix formalism.

5. Conclusion

We have determined unitary transformations, parametrized by angles θ, ϕ, which enable Q to implement a foolproof winning strategy for the quantum penny flip game. These transformations are derived using both the formalism of geometric algebra, which facilitates a geometric approach, and also density matrices. The matrix condition given by Meyer[1] for the general solution is in effect parametrized and solved by this means. Geometric algebra in general has the significant benefit of an intuitive understanding and offers better insight into quantum games and, for the quantum penny flip game, allows an analysis using operations in 3-space with real coordinates, thus permitting a visualization that is helpful in determining Q's winning strategy. A natural extension of the present work (in progress) is to apply geometric algebra to n-player quantum games, in which all players perform local quantum mechanical actions on entangled states, with the outcome determined by measurement of the final state.

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